

Constant communication complexity protocols for multiparty accumulative boolean functions

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Abstract

Generalizing a boolean function from Cleve and Buhrman [3], we consider the class of *accumulative boolean functions* of the form $f_B(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_B(x_i^1 x_i^2 \dots x_i^m)$, where $X_j = (x_1^j, x_2^j, \dots, x_n^j)$, $1 \leq j \leq m$ and $t_B(x_i^1 x_i^2 \dots x_i^m) = 1$ for input m -tuples $x_i^1 x_i^2 \dots x_i^m \in B \subseteq A \subseteq \{0, 1\}^n$, and 0, if $x_i^1 x_i^2 \dots x_i^m \in A \setminus B$. Here the set A is the input *promise* set for function f_B . The input vectors X_j , $1 \leq j \leq m$ are given to the $m \geq 3$ parties respectively, who communicate cbits in a distributed environment so that one of them (say Alice) comes up with the value of the function. We algebraically characterize entanglement assisted LOCC protocols requiring only $m - 1$ cbits of communication for such multipartite boolean functions f_B , for certain sets $B \subseteq \{0, 1\}^n$, for $m \geq 3$ parties under appropriate uniform parity promise restrictions on input m -tuples $x_i^1 x_i^2 \dots x_i^m$, $1 \leq i \leq n$. We also show that these functions can be computed using $2m - 3$ cbits in a purely classical deterministic setup. In contrast, for certain m -party accumulative boolean functions ($m \geq 2$), we characterize promise sets of mixed parity for input m -tuples so that $m - 1$ cbits of communication suffice in computing the functions in the absence of any *a priori* quantum entanglement. We compactly represent all these protocols and the corresponding input promise restrictions using uniform group theoretic and hamming distance characterizations.

Keywords: communication complexity, boolean functions, entanglement, Hamming distance

1 Introduction

The computation of a function of several variables in a distributed environment may require substantial communication between spatially separated parties; typically, different components of the input are available with the different parties, and one of the parties is required to eventually come up with the value of the function. Kremer [6] showed that computing the *two-party inner product* function $IP(X, Y) = (x_0 y_0 + x_1 y_1 + \dots + x_{n-1} y_{n-1}) \bmod 2$, requires $\Omega(n)$ qubits of communication. This result holds for the communication complexity model given by Yao [11], permitting quantum channels for communicating qubits between the two parties. The linear lower bound was already known for the scenario where only classical communication is permitted in a purely deterministic classical setting [6, 7]. In the restricted scenario as in [2, 3, 1] where no quantum communication is permitted, some saving in classical communication complexity results on exploiting *a priori quantum entanglement* and contextuality effects in *quantum measurement*. Quantum entanglement

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provides some correlation over spatially separated qubits. Buhrman, Cleve and van Dam [2], have shown that quantum entanglement can help in gaining advantage over classical communication for certain problems. One such problem is where three parties, Alice, Bob and Carol are each given two-bit vectors $X = (x_1, x_0)$, $Y = (y_1, y_0)$, $Z = (z_1, z_0)$, respectively. Alice is required to come up with the result of the evaluation of the function $h(X, Y, Z) = x_1 \oplus y_1 \oplus z_1 \oplus (x_0 \vee y_0 \vee z_0)$ given the input promise $x_0 \oplus y_0 \oplus z_0 = 0$. Buhrman et al. [2] show that two cbits of communication is sufficient for Alice to come up with the answer in the presence of three-party a priori quantum entanglement. This result was further used for computing $g(x, y, z) = \frac{(x+y+z) \bmod 4}{2}$ where x, y, z are two-bit integers and $x + y + z = 0 \pmod{2}$. It is easy to see that $g(x, y, z)$ is either 0 or 1, and, is indeed the second-least significant bit in the binary representation of $x + y + z$. It was shown that Alice can come up with the value of the function with only 2 cbits of communication (naturally, all three parties can possess the value after a total of 3 cbits of communication). The authors also established a lower bound of 4 cbits on any exact classical protocol generating the value of $g(x, y, z)$ at each of the parties.

A gap of one cbit between the classical lower bound and the entanglement assisted upper bound was also demonstrated for a three-party problem by Cleve and Burhman [3]. They worked on the three-party function $f(X, Y, Z) = (x_1 y_1 z_1 + x_2 y_2 z_2 + \dots + x_n y_n z_n) \bmod 2$; where X, Y, Z are n bit vectors given to Alice, Bob and Carol, respectively. They demonstrated that with preshared entanglement, only two classical bits of communication is required to compute f where the i th input triple $x_i y_i z_i$ is parity promise restricted to be of odd parity. They also showed that any classical protocol computing f will require at least three bits of communication. Later, Buhrman, van Dam, Hoyer and Tapp [1] considered a generalization $F(X)$ of the above mentioned function $g(x, y, z)$ of Buhrman et al. [2]. This function is a partial function $F : V^m \rightarrow \{0, 1\}$ where $V = \{0, \dots, 2^n - 1\}$. Its computation depicts a bigger gap (a logarithmic factor in the number m of parties), between entanglement assisted communication complexity and purely classical communication complexity. This function is defined as $F(X) = \frac{1}{2^{n-1}}((\sum_{i=1}^m x_i) \bmod 2^n)$, where $x_i \in V = \{0, \dots, 2^n - 1\}$ and $((\sum_{i=1}^m x_i) \bmod 2^{n-1} = 0)$. It is easy to observe that F computes the n th least significant bit of the sum of the x_i 's, which is 1 if the sum is an odd multiple of 2^{n-1} , and 0, otherwise. Although the gap is asymptotic, a logarithmic factor in m , it is still a constant for a fixed number of parties. Raz [9], demonstrated exponential communication complexity gaps for certain partial functions in Yao's model [11], where qubit communication is permitted.

The most interesting results are those of linear lower bounds on the numbers of cbits (or qubits) required for the two-party inner-product problem of computing $IP(X, Y)$ as shown by Cleve, van Dam, Nielsen and Tapp [4], even in the presence of a priori quantum entanglement. They show that such lower bounds hold for the exact problem as well as for bounded probability of failure. They use a "quantum" reduction from a quantum information theory problem to the inner product problem and use a non-trivial consequence of Holevo's theorem [8, 5] to establish the lower bound. Since quantum information subsumes classical information, this is also an alternative proof for the linear classical communication complexity lower bound for the inner product problem.

The general 3-party partial boolean function may be written as a mapping from a promise restricted subset of $\{0, 1\}^n \times \{0, 1\}^n \times \{0, 1\}^n$ into $\{0, 1\}$. In this paper, We consider 3-party functions of the form $f(X, Y, Z) = \bigoplus_{i=1}^n (l_i \wedge m_i \wedge n_i)$, where $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n)$ and $Z = (z_1, z_2, \dots, z_n)$, are boolean vectors with all the input triples $x_i y_i z_i$, $1 \leq i \leq n$ obeying uniform (either even or odd) parity promise restriction. Literals l_i, m_i, n_i represent x_i, y_i, z_i appearing in the minterm $l_i \wedge m_i \wedge n_i$, either complemented or uncomplemented. Generalizing to $m \geq 3$ parties, we consider the class of boolean functions of the form $f_B(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_B(x_i^1 x_i^2 \dots x_i^m)$, where $X_j = (x_1^j, x_2^j, \dots, x_n^j)$, $1 \leq j \leq m$ and $t_B(x_i^1 x_i^2 \dots x_i^m) = 1$ for input m -tuples $x_i^1 x_i^2 \dots x_i^m \in$

$B \subseteq A \subseteq \{0, 1\}^n$, and 0, if $x_i^1 x_i^2 \dots x_i \in A \setminus B$. Here the set A is the promise set for function f_B . We call all such boolean functions as *accumulative boolean functions*. In all these functions, the inputs given to the $m \geq 3$ parties are n -bit vectors. These accumulative boolean functions are inspired by the three party function in [3]. For one class of eight such 3-party functions, we algebraically characterize and represent protocols with preshared quantum entanglement in Section 2, where at most two cbits are communicated between the three parties, and Alice finally comes up with the value of the function (see Theorem 1). We use the *Kliens four-group* (*Vierergruppe*), V_4 , given by the matrix V_4 of Section 2 to represent the local quantum operations necessary for these functions for different combinations of (promise restricted) inputs (see Theorem 2). The promise restriction is even (or odd) that is, $x_i \oplus y_i \oplus z_i = 0(1)$, where x_i, y_i, z_i are the i th elements of the vectors X, Y, Z , respectively. for $1 \leq i \leq n$.

We generalize the results of Section 2 to $m \geq 4$ parties in Section 3 for computing accumulative boolean functions with $m - 1$ cbits of communication. More precisely, we design non-trivial promise sets and m -partite maximally entangled states using hamming distance characterizations for supporting entanglement assisted protocols (see Theorems 4 and 5 and Corollaries 1 and 2). These m -party protocols use local unitary operations such as the Hadamard operator H and the rotation operator R , as defined in Section 3. We represent the exact manner of applying these operations for the set of all accumulative boolean functions considered in this paper by the matrices N_m of Section 3. The matrix N_m is characterized in terms of the group represented by matrix V_4 of Section 2. In Section 3, we also show that $2m - 3$ cbits are sufficient to compute these functions in the deterministic classical setting with no a priori quantum entanglement. The question whether the gap between the $m - 1$ cbit entanglement assisted protocol and the $2m - 3$ purely classical protocol can be reduced remains open.

We also study the classical communication complexity of several classes of 3-party functions (in Section 4) in the absence of a priori quantum entanglement and show that two cbits of communication is sufficient for each such class of functions. The input promise restrictions in these cases are carefully chosen combinations of odd and even parities. In addition, we consider multiparty generalizations in Section 4, where m parties require $m - 1$ cbits of communication but no m -partite a priori entanglement, for computing certain mixed parity promise restricted accumulative boolean functions.

The main contribution of our work is the characterization and classification of various classes of accumulative boolean (partial) functions and the design of the appropriate input promise restrictions leading to constant communication complexity protocols; these protocols typically use $O(m)$ cbits when m parties are involved. Use of algebraic and combinatorial structures and properties help us in elegantly representing our newly defined functions and their LOCC protocols in compact notation. Throughout the paper we use the same commutative group V_4 of four elements and its higher cardinality generalizations as required in Sections 3 and 4 for multiparty accumulative boolean function evaluation. Suitable a priori tripartite or multipartite quantum entanglements are designed for the classes of functions in Sections 2 and 3 in order to design $m - 1$ cbit protocols; no quantum entanglement is needed in the case of $m - 1$ cbit protocols for the other classes of functions in Section 4.

2 Local operations for entanglement assisted protocols

Let f_u denote the accumulative boolean function $f_u(X, Y, Z) = \bigoplus_{i=1}^n t_u(x_i y_i z_i)$ defined over input boolean vectors $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n)$ and $Z = (z_1, z_2, \dots, z_n)$, with the i th input triple $x_i y_i z_i$ obeying an odd or even parity (promise) restriction. Here, $t_u(x_i y_i z_i) = l_i \wedge m_i \wedge n_i$ for

bit pattern $u = u_1u_2u_3$, such that l_i, m_i, n_i are $x_i(\neg x_i), y_i(\neg y_i), z_i(\neg z_i)$, for $u_1 = 1(0), u_2 = 1(0), u_3 = 1(0)$, respectively. We say that $t_u(x_iy_iz_i)$ is the i th minterm of type $u = u_1u_2u_3$. If $u = 011$, $t_u(x_iy_iz_i) = \neg x_i \wedge y_i \wedge z_i$. Determining $f_u(X, Y, Z)$ by computing each $t_u(x_iy_iz_i), 1 \leq i \leq n$ at Alice's site would require n cbits of communication: if Bob communicates $y_i, 1 \leq i \leq n$ to Alice, then Alice knows its own input bit x_i and can determine z_i using even parity promise given by $x_i \oplus y_i \oplus z_i = 0$. However, we wish to compute $f_u(X, Y, Z)$ using only 2 cbits of communication in an entanglement assisted protocol.

Consider the four *even parity* functions $f_u, u = 000, 011, 101, 110$. These functions are defined with input triples $x_iy_iz_i$ restricted by even parity promise set $E^3 = \{000, 011, 101, 110\}$ for each of the four bit patterns u of even parity. (For the four odd parity patterns $u = 001, 010, 100, 111$, we have four more functions f_u , which we call *odd parity* functions. These four odd functions will have input triples $x_iy_iz_i$ restricted by patterns in the odd parity promise set $O^3 = \{001, 010, 100, 111\}$). We develop protocols for the even parity functions; the treatment for the four odd parity functions is similar and symmetrical.

In the following, we first study the (0 and 1) values of $t_u(x_iy_iz_i)$ in terms of u and $x_iy_iz_i$, both belonging to the promise set E . We then design the local operations necessary on each of the three qubits, for each $1 \leq i \leq n$, finally, leading to the complete protocol. We need some notation. Let $u+, (x_iy_iz_i)+$ denote the successors of $u, x_iy_iz_i$, respectively, for values of these 3-bit patterns from the sequence $\langle 000, 011, 101, 110 \rangle$, where the successor of 110 roles back cyclically to 000. We have the following observation.

Observation 1 For all $u, x_iy_iz_i$ in the sequence $\langle 000, 011, 101, 110 \rangle$, $t_{u+}((x_iy_iz_i)+) = t_u(x_iy_iz_i)$.

Proof: Follows from the definitions of f_u and t_u . \square The value of $t_u(x_iy_iz_i), 1 \leq i \leq n$, is 1 if $u = x_iy_iz_i$, and 0, otherwise. We first show that Alice, Bob and Carol cannot come up with bits $a_i, b_i, c_i, 1 \leq i \leq n$, using deterministic classical algorithms locally, such that $t_u(x_iy_iz_i) = a_i \oplus b_i \oplus c_i$. (We consider the case where $u = 000$ but other values of u have similar analyses). If this were possible then Alice, Bob and Carol would have to come up (using classical deterministic algorithms) with boolean values $a_0(a_1), b_0(b_1)$ and $c_0(c_1)$ depending upon x_i, y_i and z_i being 0(1), respectively. Considering the four even parity patterns possible for $x_iy_iz_i$, we therefore have to satisfy (i) $a_0 \oplus b_0 \oplus c_0 = 1$, (ii) $a_0 \oplus b_1 \oplus c_1 = 0$, (i) $a_1 \oplus b_0 \oplus c_1 = 0$ and (i) $a_1 \oplus b_1 \oplus c_0 = 0$. Observe that summing up the left hand sides gives even parity whereas we have odd parity on the right hand side, a contradiction. We call such an impossibility as *classical contextuality failure (henceforth CCF)*. This may be viewed as a *non-locality game* that the three parties cannot win using any local deterministic classical strategy. In this game the parties can only do local operations but are not supposed to communicate. Using a priori tripartite quantum entanglement however, we can work out local unitary operations on the three i th qubits in the three parties so that the resulting i th entanglement on (standard basis) local measurements gives results a_i, b_i, c_i (in sites of Alice, Bob and Carol, respectively), such that $t_u(x_iy_iz_i) = a_i \oplus b_i \oplus c_i$. So, the game can be won by the three parties using a priori entanglement and local unitary operations as we develop below. Observe that with starting entanglement $|\psi_3\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$, identity operations (denoted by I) on each qubit keeps the entanglement unchanged, thereby leaving only even parity patterns of basis states on measurement, yielding eigenvalues +1. For other input triples $x_iy_iz_i \neq u$, $t_u(x_iy_iz_i)$ must be zero. So, we require to use local unitary operations on the three i th qubits in the three parties resulting in entanglements with only odd parity patterns of basis states; we note that operations IHH, HIH and HHI on $|\psi_3\rangle$ result in odd parity basis state patterns, $\frac{1}{2}(|001\rangle + |010\rangle + |111\rangle - |100\rangle)$, $\frac{1}{2}(|001\rangle + |100\rangle + |111\rangle - |010\rangle)$ and $\frac{1}{2}(|010\rangle - |001\rangle + |100\rangle + |111\rangle)$, respectively. Here H denotes the one qubit Hadamard operation, given as $H|0\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}$ and

$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$. For $x_i y_i z_i \neq u$, the measured basis state is therefore one of the four odd parity states $|001\rangle, |010\rangle, |100\rangle, |111\rangle$; the measured pattern of eigenvalues is used to set an even parity pattern abc from the patterns 110, 101, 011, 000, thereby realizing $t_u(x_i y_i z_i) = a_i \oplus b_i \oplus c_i = 0$. (Basis state $|1\rangle$ has eigenvalue -1, which we interpret as 0, and basis state $|0\rangle$ has eigenvalue 1, interpreted as 1). Symmetrically, for $x_i y_i z_i = u$, the measured basis state is one of the four even parity states $|000\rangle, |011\rangle, |101\rangle, |110\rangle$; the measured pattern of eigenvalues is used to set an odd parity pattern $a_i b_i c_i$ like 111, 100, 010, 001, thereby realizing $t_u(x_i y_i z_i) = a_i \oplus b_i \oplus c_i = 1$. It is now easy to assign local unitary operations corresponding to $t_u(x_i y_i z_i)$ as III, IHH, HII, HHI for $x_i y_i z_i = 000, 011, 101, 110$, respectively, if $u = 000$. Each agent can determine whether to apply I or H to its own i th qubit depending on its i th input bit. This gives the first row in Table 1. For the other rows we can very well choose the local operations to be III in the diagonal and IHH, HHI and HII for $u \oplus x_i y_i z_i$ values 011, 110 and 101, respectively, thereby giving the local operations corresponding to $t_u(x_i y_i z_i)$ (see Observation 1). This completes Table 1. Note that each of IHH, HHI and HII give only odd parity basis states in the resulting entanglement, ensuring correct evaluation of $t_u(x_i y_i z_i) = 0$ for $x_i y_i z_i \neq u$, as already explained above. For each i , the local operations generate a_i, b_i and c_i such that $t_u(x_i y_i z_i) = a_i \oplus b_i \oplus c_i$. Bob and Carol can communicate b_i and c_i to Alice for each $1 \leq i \leq n$, totalling $2n$ cbits, so that Alice can compute $t_u(x_i y_i z_i)$ for each i (and hence Alice computes $f_u(X, Y, Z)$). However, Bob (and Carol) may very well compute the XOR of his (her) respective n bits b_i (respectively, c_i), $1 \leq i \leq n$, and finally communicate just one cbit to Alice for determining $f_u(X, Y, Z)$, totalling only 2 cbits of communication. Now we have the entire set of protocols for each of the four even parity functions f_u . We summarize our result in the following theorem.

Theorem 1 *The protocols for computing f_u using only two cbits of communication are realized using local unitary operations I and H as given in Table 1 and using n sets of a priori tripartite entanglement states $|\psi_3\rangle$.*

It is not difficult to verify that a similar and symmetrical result holds also for odd parity functions f_u , where $u \in \{001, 010, 100, 111\}$.

2.1 An algebraic representation for local operations

Now we study some algebraic properties of local operations for f_u , in terms of recursively defined groups. This group theoretic study is motivated by the intricate but interesting patterns in Table 1.

Definition 1 *We define matrices M_i and M'_i recursively as follows.*

1. $M_1 = I$ and $M'_1 = H$.
2. $M_{i+1} = \begin{pmatrix} IXM_i & HXM'_i \\ HXM'_i & IXM_i \end{pmatrix}$
3. $M'_{i+1} = \begin{pmatrix} IXM'_i & HXM_i \\ HXM_i & IXM'_i \end{pmatrix}$

In the above definition, AXB denotes tensor multiplication of each element of the matrix B by the element or entity A . M_3 is precisely the matrix of local operations as in Table 1 corresponding to terms $t_u(x_i y_i z_i)$ for functions f_u . Using bit triples $a = 000, b = 011, c = 101, d = 110$ for III, IHH, HII and HHI, respectively, consider the group represented by the matrix V_4 below, where

the rows (columns) are indexed from left to right (top to bottom) by group elements a, b, c, d , in that order. The group we require for representing the local unitary operations for $t_u(x_i y_i z_i)$, for all $u \in \{000, 011, 101, 110\}$ and all $1 \leq i \leq n$, (and therefore, for f_u) is given by the matrix

$$V_4 = \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}$$

where the $(u, x_i y_i z_i)$ th element in the matrix V_4 is the element $u.x_i y_i z_i$ in the group represented by matrix V_4 . Here, we could imagine $a = 000, b = 011, c = 101, d = 110$ for even parity functions and $a = 001, b = 010, c = 100, d = 111$ for odd parity functions.

Theorem 2 *The local operations corresponding to $t_u(x_i y_i z_i)$, as depicted in Table 1 and matrix M_3 , are represented by the group element $u \odot v$ in the group represented by the matrix V_4 , where $v = x_i y_i z_i$ and \odot represents the group operation.*

We call the above matrix M_3 represented as V_4 , the *game matrix* for the 3-party case. Note also that each entry in M_m (M'_m) has an even (odd) number of H operations. We use this property in Sections 3 and 4. The following lemma states a useful property of matrices M_m and M'_m . This property is at the heart of the multipartite protocols designed in subsequent sections.

Lemma 1 *Let m be the number of parties, n be the size of the input bit vector given to each party and u be an m -bit string of even parity. The local operations matrix with entries corresponding to $u \oplus p_i$ indexed by u in the rows and p_i in the columns is identical to the matrix M_m (M'_m), where $p_i = x_i^1 x_i^2 \dots x_i^m$ is the i th input m -tuple, $1 \leq i \leq n$, of even (odd) parity.*

Proof: Proof follows by induction, using the definitions of M_m and M'_m . \square

2.2 Correlation preserving reducibilities

We now know that all functions f_u can be computed with n sets of a priori tripartite entanglements and promise constrained n -bit vector inputs to Alice, Bob and Carol, with only 2 cbits of communication. In Table 2, we show how we may simulate each function in this set of eight functions by any of the other seven. The simple trick is to toggle all bits of one or more of the three input vectors and accordingly choose the simulating function; the promise automatically gets set as required in the simulations. (When bits of an odd number of vectors are toggled, the parity must switch). This equivalence also implies (following the lower bound proof in Cleve et al. [3]), that each of these eight functions has a classical computation protocol with 3 cbits of communication. In addition, this equivalence also implies that none of these functions can be computed using 2 cbits of communication. We summarize these facts in the following theorem.

Theorem 3 *Each of the eight functions f_u can be computed by a classical protocol that requires only three cbits of communication. Moreover, none of these functions has a two cbits classical communication protocol.*

The above simulation of one function by any of the seven other functions is done using reductions that do not alter correlations between bit vectors given to the three parties. We call such reductions as *correlation preserving reductions*.

3 Hamming distance characterizations of promise sets

In this section we extend entanglement assisted protocols requiring constant classical communication complexity, to accumulative boolean functions for $m \geq 4$ parties. Extending the protocols of Section 2 essentially means spelling out local operations in each of the m parties; we do this by using the matrix M_m of Section 2. We state the required definitions and notation. Let E^m (O^m) denote the set of 2^{m-1} even (odd) parity m -bit strings. We denote the (even parity) functions as $f_u(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_u(x_i^1 x_i^2 \dots x_i^m)$, where $u \in E^m$, and $t_u(x_i^1 x_i^2 \dots x_i^m)$ is 1 for $x_i^1 x_i^2 \dots x_i^m = u \in A \subseteq E^m$, and 0, otherwise. (A similar and symmetric definition is possible for odd parity functions). Here, the set A is the input promise set to which the input bit strings $x_i^1 x_i^2 \dots x_i^m$, $1 \leq i \leq n$, are restricted. We characterize certain promise subsets $A \subseteq E^m$, permitting entanglement assisted protocols using exactly $m - 1$ cbits of communication, using n sets of m -partite maximally entangled states, and local unitary operations governed by matrix M_m . For $m = 4$, we show that the permissible promise sets are $A \subseteq E^4 \setminus \{x \mid \text{where } (u \oplus x) = 1111, x \in E^4\}$. So, for $f_{0001}(X_1, X_2, X_3, X_4)$, the promise sets that work are $A \subseteq E^4 \setminus \{1110\}$, with a unique entangled state that we develop below; this entangled state contains the eight odd parity basis states. Finally, we also consider cases where $m \geq 5$. For these generalized mutiparty cases, we define accumulative boolean functions $f_B(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_B(x_i^1 x_i^2 \dots x_i^m)$, where we define $t_B(x_i^1 x_i^2 \dots x_i^m) = 1$ for input m -tuples $x_i^1 x_i^2 \dots x_i^m \in B \subseteq A \subseteq O^m$, and 0, otherwise. Here the set A is the promise set for function f_B .

3.1 Promise sets and entangled states

Restricting m -party local operations to those defined by matrices M_m , we first establish a few results correlating choices of superposition patterns that use all the odd (or even) parity basis states in maximal m -partite entanglement states. In particular, we would be considering local operations as given in M_m and entanglement state $|\psi_m\rangle = \frac{1}{2^{(m-1)/2}} \sum_{v \in O^m} (-1)^{sg(v)} |v\rangle$, where $sg(v) = 1$ only for superposition basis states carrying minus sign, and $sg(v) = 0$, otherwise. We derive a suitable functions sg for our protocols below. We need some notation. Let $|s\rangle = |s_1 s_2 \dots s_m\rangle$ denote an m -partite standard basis state in O^m in the 2^m -dimensional Hilbert space $H^{\otimes m}$. Let $|s_{ij}\rangle$, $1 \leq i < j \leq m$, denote the (sub)state of $|s\rangle$ in the 2^{m-2} -dimensional Hilbert space $H^{\otimes(m-2)}$ with the qubits of the i th and the j th parties in $|\psi_m\rangle$ dropped. We use the notation $H_i H_j$ to denote the operator where local Hadamard operations are performed on the i th and j th qubits in the respective sites and the identity operation is performed on all other qubits. First we establish the following result.

Lemma 2 *Given two basis states $|s\rangle$ and $|t\rangle$ in O^m , separated by hamming distance two, let $|s_{ij}\rangle = |t_{ij}\rangle$, for some $1 \leq i < j \leq m$. Then, $H_i \otimes H_j |\psi_m\rangle$ will get only even parity m -partite basis state superpositions if we set $sg(s)$ and $sg(t)$ such that $s_i \oplus s_j \oplus sg(s) \oplus sg(t) = 1$.*

Proof: It is easy to see that $H^{\otimes 2}|01\rangle$ and $H^{\otimes 2}|10\rangle$ have $|01\rangle$ and $|10\rangle$ states with opposite signs. Moreover, $|s\rangle$ and $|t\rangle$ have hamming distance two, with bit disagreement only at the i th and j th positions. So, in case (i) if $s_i \oplus s_j = 1$ (and therefore $|s_{ij}\rangle$ and $|t_{ij}\rangle$ have even parity), we assign identical signs $sg(s) = sg(t)$. Likewise, in case (ii) if $s_i \oplus s_j = 0$ (and therefore $|s_{ij}\rangle$ and $|t_{ij}\rangle$ have odd parity), we assign opposite signs $sg(s)$ and $sg(t) = 1 \oplus sg(s)$. Such assignments for function sg would ensure cancellation of all odd parity basis states. \square

For instance, consider f_{0001} (without loss of generality). Consider basis states $a = |0001\rangle$ and $b = |0010\rangle$ superimposed in the shared a priori entangled state $|\psi_m\rangle$, where $m = 4$. Considering input quadruple 0001, the entanglement remains unchanged due to operations $I^{\otimes 4}$; so, standard

basis measurements at the four sites will result in odd parity basis state patterns. Whereas for input quadruple 0010, matrix M_4 shows that we need to do H operations on the third and fourth qubits and no operations on the first two qubits. The IIHH operation on basis states $a = |0001\rangle$ and $b = |0010\rangle$ will lead to cancellation of all odd parity basis states $|0010\rangle$ and $|0001\rangle$ if a and b have the same probability amplitude with identical \pm signs (as stated in Lemma 2). Considering the same input quadruple 0010 again, and applying Lemma 2, we see that we must also give same signs for the pair of basis states $(g = |1101\rangle, h = |1110\rangle)$, but different signs for the pairs $(c = |0100\rangle, d = |0111\rangle)$, and $(e = |1000\rangle, f = |1011\rangle)$. Assigning such signs will ensure that the resulting 4-partite entangled state will give odd parity basis states on standard basis measurements at four sites. Similarly, considering five more 4-bit input quadruples 0100, 0111, 1000, 1011 and 1101, we can deduce applying Lemma 2 that basis states' pairs which must agree on their signs are respectively, (a, c) and (f, h) , (b, c) and (f, g) , (a, e) and (d, h) , (b, e) and (d, g) , and finally, (c, e) and (d, f) , whereas, basis states' pairs which must disagree on their signs are respectively, (b, d) and (e, g) , (a, d) and (e, h) , (b, f) and (c, g) , and finally, (a, g) and (b, h) . With some thought, it follows that the unique solution is to assign the same sign to basis states a, b, c, e and just the opposite sign to basis states d, f, g, h . Since we have considered the function f_{0001} , the input quadruples considered were in the promise set $O^4 \setminus \{1110\}$. Generalizing over all $u \in O^4$, we can now state the following results, where $u \oplus u' = 1111$.

Theorem 4 *Let $u \in O^4$ and $u \oplus u' = 1111$. Let the input quadruples $x_i^1 x_i^2 x_i^3 x_i^4, 1 \leq i \leq n$ be restricted to elements of any promise set $A \subseteq O^4 \setminus \{u'\}$. Using n instances of the entangled state $|\psi_4\rangle$, and local operations as in matrix M_4 , it is possible for Alice to come up with the value of $f_u(X_1, X_2, X_3, X_4) = \bigoplus_{i=1}^n t_u(x_i^1 x_i^2 x_i^3 x_i^4)$, with only three cbits of communication.*

Corollary 1 *Let $u \in O^4$ and $u \oplus u' = 1111$. Let the input quadruples $x_i^1 x_i^2 x_i^3 x_i^4, 1 \leq i \leq n$ be restricted to elements of any promise set $A \subseteq O^4$. Using n instances of entangled state $|\psi_4\rangle$, and local operations as in matrix M_4 , it is possible for Alice to come up with the value of $f_{\{u, u'\}}(X_1, X_2, X_3, X_4) = \bigoplus_{i=1}^n (t_u(x_i^1 x_i^2 x_i^3 x_i^4) \oplus t_{u'}(x_i^1 x_i^2 x_i^3 x_i^4))$, for any $u \in O^4$, with only three cbits of communication.*

Proof: It turns out that $H^{\otimes 4}$ operating on $|\psi_4\rangle$ (for input u') yields an entanglement state with only odd parity basis states. The effect is same as that with operations $I^{\otimes 4}$ for input u . So, clubbing u and u' together for $f_{u, u'}$ using minterms t_u and $t_{u'}$ does the needful. \square

It is interesting to note that we chose to assign plus and minus signs in such a manner to the basis states in the maximal entanglement $|\psi_4\rangle$ that the basis states with the same number of 1's got identical signs. This also holds for the tripartite entanglement $\frac{1}{2}(|001\rangle + |010\rangle + |100\rangle - |111\rangle)$ used by Cleve et al. [3], in their entanglement assisted protocol for computing $f_{111}(X, Y, Z) = \bigoplus_{i=1}^n x_i \wedge y_i \wedge z_i$, with only 2 cbits of communication and odd parity promise over input triples $x_i y_i z_i, 1 \leq i \leq n$.

3.2 Promise sets for the general case of multiple parties

For the multiparty accumulative boolean functions, we now pose the general version of the *non-locality game*, whose 3-party version was analyzed in Section 2. In this game, we require the j th of the m parties to receive its respective input bit x_i^j and come up with boolean value a_i^j such that $t_u(x_i^1 x_i^2 \dots x_i^m) = a_i^1 \oplus a_i^2 \dots \oplus a_i^m$. This is not possible in a purely deterministic classical setup but possible when a priori multipartite entanglement is used. Note that the parties cannot communicate in this game but may perform local operations.

So far we considered using only local operations I and H in our protocols. We now consider use of operators H and a rotation operator R defined as $R|0\rangle = |0\rangle$ and $R|1\rangle = e^{\frac{i\pi}{2}}|1\rangle$. Let N_m be the matrix obtained from matrix M_m by replacing (i) I with H and (ii) H with HR. Let $|\psi_m^{GHZ}\rangle = \frac{|0^m\rangle + |1^m\rangle}{\sqrt{2}}$ be the maximally entangled m -partite GHZ state (also called the m -CAT state). We establish the following results.

Lemma 3 *Let the i th input triple be $v = x_i^1 x_i^2 \dots x_i^m$, where the j th party is given bit x_i^j , $1 \leq j \leq m$. Suppose the j th party, $1 \leq j \leq m$, performs an H (HR) operation provided x_i^j is equal (not equal) to the j th bit of u . Then, it is possible for the j th party to come up with bit a_i^j such that $t_u(x_i^1, x_i^2 \dots x_i^m) = a_i^1 \oplus a_i^2 \dots \oplus a_i^m$.*

Proof: If the i th input triple $v = x_i^1 x_i^2 \dots x_i^m$ is identical to u , we simply perform $H^m |\psi_m^{GHZ}\rangle$, giving only even parity basis states in the resulting entanglement. For $v \in O^m$ such that v and u have hamming distance equal to an odd multiple (say $2k$, where k is odd) of 2, we observe that local operation HR is performed at $2k$ locations. This results in a local phase factor of $(e^{\frac{i\pi}{2}})^{2k} = e^{ki\pi} = -1$ for the second term in $|\psi_m^{GHZ}\rangle$, flipping its sign. With the H operations now at all the m sites, the resulting entangled state has only the odd parity basis states. So, after performing standard basis measurements at the m sites, the measured values of local qubits can be represented at their respective sites as boolean values a_i^j , such that $a_i^1 \oplus a_i^2 \dots \oplus a_i^m$ is of odd parity if and only if $v = u$. Thus, we have a_i^j generated at the j th site such that $t_u(x_i^1, x_i^2 \dots x_i^m) = a_i^1 \oplus a_i^2 \dots \oplus a_i^m$, winning the non-locality game. \square

Theorem 5 *Let $u \in O^m$. Let the input m -tuples $x_i^1 x_i^2 \dots x_i^m$, $1 \leq i \leq n$, be restricted to the elements of any promise set $A \subseteq \{v | v \in O^m, \text{ and either } v = u \text{ or } v \oplus u \text{ has parity equal to an odd multiple of } 2\}$. Using n instances of entangled state $|\psi_m^{GHZ}\rangle$, and local operations as in matrix N_m , it is possible for Alice to come up with the value of $f_u(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_u(x_i^1 x_i^2 \dots x_i^m)$, with only $m - 1$ cbits of communication.*

Proof: Computing $f_u(X_1, X_2, \dots, X_m)$ requires evaluating the XOR of terms $t_u(x_i^1 x_i^2 \dots x_i^m)$, where each term can be written as $a_i^1 \oplus a_i^2 \dots \oplus a_i^m$, $1 \leq i \leq n$, as shown in Lemma 3. We can compute A_j , the XOR of a_i^j , $1 \leq i \leq n$ in the j th party locally, for each $1 \leq j \leq m$. Then, using $m - 1$ cbits of communication, the bits A_j , $2 \leq j \leq m$, can be communicated to the first party for evaluation of $f_u(X_1, X_2, \dots, X_m)$. \square

Corollary 2 *Let $u \in O^m$. Let $B \subseteq O^m$ be any set of elements $v \in O^m$ (including u), such that u and v have hamming distance equal to an even multiple of 2. Let the input m -tuples $x_i^1 x_i^2 \dots x_i^m$, $1 \leq i \leq n$ be restricted to the elements of any promise set $A \subseteq O^m$. Using n instances of entangled state $|\psi_m^{GHZ}\rangle$, and local operations as in matrix N_m , it is possible for Alice to come up with the value of $f_B(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_B(x_i^1 x_i^2 \dots x_i^m)$, with only $m - 1$ cbits of communication.*

Proof: The promise set $A \subseteq O^m$ is arbitrary here. For elements of B , note that the phase term of $(e^{\frac{i\pi}{2}})^{2k} = e^{ki\pi} = +1$ for the second term in $|\psi_m^{GHZ}\rangle$, leaving its sign intact because k is even. With the H operations now at the m sites, the resulting entangled state has only the even parity basis states, like what happens when we apply $I^{\otimes m}$ to $|\psi_m^{GHZ}\rangle$. So, we can club u and the entire set B together as distinguished from the rest of the elements of O^m . Hence we can compute f_B using $m - 1$ cbits of communication by correctly determining the parity $t_B(x_i^1 x_i^2 \dots x_i^m)$, $1 \leq i \leq n$. \square

We end this section with a classical protocol scheme for computing such promise restricted functions by observing that all the functions $f_u, u \in \{0, 1\}^m$, are mutually reducible as depicted in

Table 1. So, it suffices to deal with f_{0^m} . The promise set comprises even parity m -tuple patterns that have 2, 6, ..., $2(2k-1)$, ... 1's in the pattern, contributing 0's to the function, and the pattern 0^m contributing 1. Alice therefore needs to determine $(n-p) \bmod 2$, where p is number of non-zero m -tuples; it is easy to see that p is half the modulo 4 sum of the total number of non-zero bits given as inputs to the m parties. So, the $m-1$ parties can compute the modulo 4 sum of non-zero bits in their respective input vectors and pass on the two bits to Alice, whence she can compute p and $(n-p) \bmod 2$ as the value of the function f_{0^m} . This results in classical communication complexity $2m-2$ cbits, which can further be reduced by one bit since we know that input m -tuples are of even parity.

4 Classical protocols for accumulative boolean functions with mixed parity promise

Unlike functions f_u where the promise was strictly based on either even or odd parity, we now consider new classes of functions where input triples are restricted by various mixed parity constraints. We characterize (i) these promise sets, and (ii) the LOCC protocols for computing these accumulative boolean functions with (a constant number of) $m-1$ cbits where $m \geq 3$ is the number of parties. Let $g_u(X, Y, Z) = \bigoplus_{i=1}^n t_u(x_i y_i z_i)$. Here, as in the case of f_u , $t_u(x_i y_i z_i)$ is again a minterm determined by the bit pattern u . First consider $u = 000$ where we restrict input $x_i y_i z_i$ to the elements of the set $O = \{000, 001, 010, 100, 111\}$. Note that in this case we have a mix of even and odd parities, with 000 coming along with all the four odd parity patterns. With the same four odd parity patterns, we can define three more functions g_u , where $u = 011, 101, 110$; the input patterns in the promise set being $\{u, 001, 010, 101, 111\}$. We reiterate that $t_u(i) = 1$ if and only if $u = x_i y_i z_i$, very much as in the case of functions f_u .

4.1 Protocols for inputs with mixed parity promise

Now we follow the design technique similar to the one in the previous sections for coming up with protocols for Alice computing g_u for input vectors X, Y, Z given to Alice, Bob and Carol, respectively, obeying promise restrictions as just mentioned. Note that the pattern u has even parity. So, for $t_u(u)$, we may very well settle with an even number (may be none) of toggling local operations over $x_i y_i z_i = u$, keeping the inter-party parity over the input $x_i y_i z_i$ unchanged even after toggling. This is indeed possible if we start with a bit pattern $S_i = u$ for the i th triple, where one bit of S_i is in each party, and we toggle the respective bits in each party if the XOR of the input triple bit x_i, y_i or z_i , for the respective party, with the respective bit in u is 1, and, do nothing otherwise. Since, $t_u(x_i y_i z_i) = 0$ for $x_i y_i z_i \neq u$, we require to get a zero contribution in such cases; due to odd hamming distance between the promise permitted odd input parity triples $x_i y_i z_i \neq u$, and the even parity of u , only an odd number of toggling operations can result in toggling operations controlled by the odd parity pattern $u \oplus x_i y_i z_i$. Since we start with even parity $S_i = u$, this action will result in S_i gaining odd parity only for input triples $x_i y_i z_i$ of odd parity. This holds for any even parity pattern u (and, therefore for all functions g_u for even parity patterns u). Once this step is over, we observe that if the i th input triple is u , then S_i will result in an even parity patterns; otherwise, S_i will end up with odd parity. Naturally, toggling all bits in all S_i now will result in odd parity patterns for input triple u and even parity for others. Indeed, all we need to do at this stage is to compute XOR of all $3n$ bits of $S_i, 1 \leq i \leq n$, yielding $g_u(X, Y, Z)$. The rest of the protocol is identical to the remaining steps of protocols for f_u . In particular, local XOR over three n -bit vectors is used before the two parties Bob and Carol communicate one bit each

to Alice. The only difference is that we use pattern $S_i = u$, which is a classical state (say, 000 for $u = 000$), and our local operations were simply toggling classically 0 and 1 states. We summarize this fact in the following theorem; we also generalize this result in Section 4.2 to an $m - 1$ cbits classical protocol for the m -party versions of such mixed parity functions using the group M_m and M'_m of Section 2.

Theorem 6 *The eight three-party functions g_u of mixed parity promise set $O^3 \cup \{u\}$ can be computed where Alice comes up with the value of the function, using only two cbit distributed protocols requiring no a priori quantum entanglement and only local classical operations.*

4.2 Extension to multiminterm functions

Extending the above ideas, note that we may increase the cardinality of the promise sets by adding other even parity patterns in addition to u for g_u , giving rise to new functions say $g_{uu'}$, where only input triples $x_i y_i z_i$ of even parity result in the value 1 for $t_{uu'}(x_i y_i z_i)$, and therefore the i th term $t_{uu'}(x_i, y_i, z_i)$ may be viewed as a multiminterm boolean expression with XOR (or OR) operation between them. For instance, with $u = 000$ and $u' = 101$, we have $g_{uu'}(X, Y, Z) = \bigoplus_{i=1}^n ((\neg x_i \wedge \neg y_i \wedge \neg z_i) \oplus (x_i \wedge \neg y_i \wedge z_i))$. We can have six choices of u, u' , combinations without repetitions of four patterns from the set E taken two at a time. Likewise, for three minterms, we will have four functions, and only one function if we take all four even parity minterms. Result similar to Theorem 6 holds for all these functions. In summary, the promise sets for these functions are $A \cup O^3$, where $A \subseteq E^3$, and O^3 and E^3 are the sets of three bit odd and even parity patterns, respectively, as defined in Section 2. The protocol remains similar to the one corresponding to Theorem 6 for the computation of functions g_u . We summarize the result as a corollary.

Corollary 3 *Consider three party accumulative boolean functions $g_A(X, Y, Z) = \bigoplus_{i=1}^n t_A(x_i y_i z_i)$, where (i) $A \subseteq E^3$, (ii) the input promise restricts $x_i y_i z_i \in A \cup O^3$, and (iii) $t_A(x_i y_i z_i)$ is 1 for $x_i y_i z_i \in A$, and 0, otherwise. The functions g_A of such mixed parity promise can be computed with Alice coming up with the function value with only two cbit distributed protocols requiring no a priori quantum entanglement and only local classical operations.*

Generalization of the basic result in Theorem 6 to multiple parties is as follows. The function $g_A(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_A(x_i^1 x_i^2 \dots x_i^m)$, where $t_A(x_i^1 x_i^2 \dots x_i^m)$ is 1 for $x_i^1 x_i^2 \dots x_i^m \in A \subseteq E^m$, and 0, otherwise. The input promise set is $A \cup O^m$ as in Theorem 3.

For odd number $m \geq 5$, we start with any even parity bit vectors $S_i = s_i^1 s_i^2 \dots s_i^m, 1 \leq i \leq n$, where bit s_i^j is locally held by the j th party. For function g_A , where $A \subseteq E^m$, the j th party toggles s_i^j if and only if the j th bit in $u \oplus x_i^1 x_i^2 \dots x_i^m$ is 1. Here, the first party is Alice and the input vector to the j th party is $X_j = x_1^j x_2^j \dots x_n^j$. Clearly, S_i attains odd parity only for odd parity input m -tuples $x_i^1 x_i^2 \dots x_i^m$. Since m is odd, we can toggle all bits of S_i locally to get even parity S_i for even parity input m -tuples $x_i^1 x_i^2 \dots x_i^m$. For computing $g_A(X_1, X_2, \dots, X_m)$, we now need to perform XOR over all bits of all $S_i, 1 \leq i \leq n$. This can be achieved by doing XOR universally over all s_i^j locally at the j th site for $i \leq i \leq n$, and then communicating these results to Alice from each party $j, 2 \leq j \leq m$, using a total of m cbits. Alice can then do the obvious rest.

When $m \geq 4$ is even, all we need to do is choose odd parity $S_i, 1 \leq i \leq n$, to begin with. It is easy to now check that the rest of the steps are similar to the case where m is odd, and we also do not need the final universal toggling step over all bits of all S_i . The result is summarized as follows.

Theorem 7 *Consider the multipartite accumulative boolean functions $g_A(X_1, X_2, \dots, X_m) = \bigoplus_{i=1}^n t_A(x_i^1 x_i^2 \dots x_i^m)$ with j th party getting input bit vector $X_j, 1 \leq j \leq m$, where $t_A(x_i^1 x_i^2 \dots x_i^m)$ is*

1 for $x_i^1 x_i^2 \dots x_i^m \in A \subseteq E^m$, and 0, otherwise. The input promise restricts $x_i^1 x_i^2 \dots x_i^m \in A \cup O^m$. The multiparty functions g_A of such mixed parity promise can be computed with Alice coming up with the function value with only $m - 1$ cbit distributed protocols requiring no a priori quantum entanglement and only local classical operations.

5 Concluding remarks

We investigated different types of promise sets for input tuples for evaluating accumulative boolean functions with constant communication complexity. We demonstrated purely classical m -party protocols requiring $m - 1$ cbits of communication for mixed parity promise sets where tuples of opposing parities (even and odd) contribute 1's and 0's respectively, to the accumulative boolean function (see Section 4). Here, one or more m -tuples of even (odd) parity may be permitted, contributing 1's, just as multiple non-contributing m -tuples of the opposite parity are permitted as inputs, contributing 0's. For input promise sets containing only even (or odd) parity bit patterns, we designed constant communication complexity entanglement assisted protocols for such accumulative boolean functions with $m - 1$ cbits for the m -party case (see Section 2 and 3). Here, discrimination is made between a specific even (or odd) parity m -bit input string (or a suitably defined subset of input strings) against all the input strings from a specific promise subset of the remaining m -bit input strings of the same parity. We designed the requisite maximally entangled states using the eight basis states of odd parity, by assigning real probability amplitudes of equal magnitudes to the basis states for the 4-party case. The signs of these amplitudes had to be chosen carefully in accordance with the chosen promise sets, as characterized in Section 3. For the general m -party problem, $m \geq 3$, we designed an alternative entanglement assisted protocol in Section 3, using $m - 1$ cbits of communication and n copies of the m -CAT entangled state.

We are currently investigating along similar lines, looking for more such algebraic structures and characterizations. Swain [10] reports constant communication complexity protocols for a class of accumulative m -party boolean functions that compute *total disagreement parity*, of (say) Alice with the rest of the parties over multiple m -tuples. The promise sets are suitably defined in a different manner; the a priori entanglements and local operation matrices too are different from what we use in this paper.

It is worthwhile unifying the protocols in this paper in terms of the patterns of local operations performed in each of the parties; it is indeed possible to find a common line in all the protocols in this paper based on the group represented by matrix V_4 . Consider Theorems 2, 4, 5, 6 and 7, and Corollaries 1, 2 and 3. Observe the functions g_u and g_A in Section 4, where a non-trivial local (toggling) operation is done on the j th bit of S_i based on the j th bit in the pattern $u \oplus p_i$, where $p_i = x_i^1 x_i^2 \dots x_i^m$ is the i th input m -tuple, $1 \leq i \leq n$. If this bit is 0, the identity operation I is done. If this bit is 1, the toggling operation is done. The local operations comprise an even number of toggling operations corresponding to $u \oplus p_i$ indexed by u in the rows and p_i in the columns, where u as well as p_i are of even parity. Symmetrically, local operations corresponding to $u \oplus p_i$ has an odd number of toggling operations, where u has even parity and p_i has odd parity. In the case of functions g_u and g_A , if p_i is even, the even parity in S_i is not disturbed by any local operation pattern, whereas, for odd parity p_i , the parity in S_i must be reversed. In this manner the protocols in Section 4 correctly compute the partial functions g_u and g_A with mixed parity promise sets. It is easy to see that the matrix M_m (M'_m) compactly encodes the XOR operation in its entry corresponding to the pattern $u \oplus p_i$ in the u th row and p_i th column, if p_i and u agree (differ) in parity. (The matrix entry must be translated replacing I by 0 and H by 1). In Section 3 we used matrix N_m derived from M_m by replacing I by H and H by HR; the matrix N_m models

local operation patterns for the m -party entanglement assisted protocols. Separately, matrices M_3 and M_4 are used in similar fashion in Sections 2 and 3. Although the two categories of problems and their protocols differed, one yielding to entanglement assistance and the other succumbing to classical means with no entanglement whatsoever, the unifying aspect was the common or similar pattern of local operations. Local operations are compactly represented by the elegant recursively defined matrices M_m and M'_m . These matrices are based on the four element group represented by the matrix V_4 (see Section 2). To the best of our knowledge, the matrices M_m and M'_m do not appear in the literature. We feel that these matrices or similar recursively defined structures may be useful in compactly representing local operations for quantum entanglement assisted protocols for other classes of problems too.

Before concluding, we also consider the two-party scenario for mixed parity promise restricted functions. In contrast to the celebrated linear lower bound on the deterministic classical communication complexity of the two-party INNER PRODUCT function (see [6, 7]), the following function g_{11} with mixed parity promise has a one cbit classical protocol. Following uniform notation, we define $g_{11}(X, Y) = \bigoplus_{i=1}^n x_i \wedge y_i$, where $x_i y_i$ is restricted to be from the mixed parity promise set $\{11, 01, 10\}$. We construct a deterministic classical one-cbit protocol where Alice and Bob first come up with bits a and b , respectively, so that $x_i \wedge y_i = a \oplus b$. Let a be called a_0 if $x_i = 0$, and a_1 , otherwise. Likewise, let b be called b_0 if $x_i = 0$, and b_1 , otherwise. We must have (i) $a_1 \oplus b_1 = 1$, (ii) $a_0 \oplus b_1 = 0$, and (iii) $a_1 \oplus b_0 = 0$. Note that assigning 1 to a_1 and b_0 , and 0 to a_0 and b_1 achieves our purpose, leading to a 1-cbit classical protocol for Alice coming up with the value of $g_{11}(X, Y)$. Indeed, we assert that the result analogous to Theorem 7 holds also for the two-party case.

We have studied only one-round, constant communication complexity protocols. We propose that problems yielding to multiple rounds be investigated and characterized. We believe that such low communication complexity problems for various input promise sets would be very useful in VLSI design and also in mobile distributed computing. We have presented results pertaining only to deterministic computations. A natural research direction is the study of probabilistic computations of partial boolean functions requiring constant or low communication complexity. Another important problem is that of settling the optimal classical communication complexity bound for the m -party (partial) functions in Section 3.

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			Local operations on qubits of i th entanglement for i th input triple as below			
Function	Promise	Apriori entanglement	000	011	101	110
f_{000}	$x_i \oplus y_i \oplus z_i = 0$	$1/2(000\rangle - 011\rangle - 101\rangle - 110\rangle)$	III	IHH	HIH	HHI
f_{011}	-do-	-do-	IHH	III	HHI	HIH
f_{101}	-do-	-do-	HIH	HHI	III	IHH
f_{110}	-do-	-do-	HHI	HIH	IHH	III

Table 1: Local operations for functions f_u for evaluating a single minterm t_u

	f_{000}	f_{001}	f_{010}	f_{011}	f_{100}	f_{101}	f_{110}	f_{111}
f_{000}		z_i	y_i	y_i, z_i	x_i	x_i, z_i	x_i, y_i	x_i, y_i, z_i
f_{001}	z_i		y_i, z_i	y_i	x_i, z_i	x_i	x_i, y_i, z_i	x_i, y_i
f_{010}	y_i	y_i, z_i		z_i	x_i, y_i	x_i, y_i, z_i	x_i	x_i, z_i
f_{011}	y_i, z_i	y_i	z_i		x_i, y_i, z_i	x_i, y_i	x_i, z_i	x_i
f_{100}	z_i	x_i, z_i	x_i, y_i	x_i, y_i, z_i		z_i	y_i	y_i, z_i
f_{101}	x_i, z_i	x_i	x_i, y_i, z_i	x_i, y_i	z_i		y_i, z_i	y_i
f_{110}	x_i, y_i	x_i, y_i, z_i	x_i	x_i, z_i	y_i	y_i, z_i		z_i
f_{111}	x_i, y_i, z_i	x_i, y_i	x_i, z_i	x_i	y_i, z_i	y_i	z_i	

Table 2: Reducibility between functions